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LETTER TO THE EDITOR

Large-cell renormalisation and systems of dimensionality larger than the upper marginal dimension

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Abstract. A recent argument dismissing the applicability of large-cell renormalisation schemes to systems of dimensionality d larger than the upper marginal dimension d_c is critically discussed. In this connection, new large-cell renormalisation results for the random walk for $d=3$ and 4 are presented which indicate convergence to the correct results.

A recent argument by Sahimi and Jerauld (1983) on the inapplicability of large-cell renormalisation for $d > d_c$ appears not to be limited to the random walk but to include 'critical' or scale invariant properties of all systems. They argued: (a) large cell renormalisation with the extrapolation of cell size $b \to \infty$ is equivalent to finite-size scaling; (b) finite-size scaling fails for $d > d_c$; and thus large-cell renormalisation must necessarily fail for $d > d_c$. Specifically, they discussed the Flory exponent ν of the (unconstrained) random walks in d dimensions and cited as supporting evidence: (c) the trend of the cell-renormalisation estimates of ν decreasing away from the exact value of $\frac{1}{2}$ as b increases in $d=3$ and 4.

I shall start with the discussion of statement (b). To support the statement, Sahimi and Jerauld cite Brézin (1982) and a singularity in the scaling *function* found by him. However, it is important to understand exactly what *aspect* of finite-size scaling fails for $d > d_c$ because of the onset of such a singularity. In particular, we note that the said singularity is now well known to be due to a 'dangerous' irrelevant variable (Fisher 1973), and a generalised scaling form remains valid.

To be specific let us consider the n-vector model where $d_c = 4$. Thus allowing for an irrelevant variable u, we have a general scaling statement (Fisher 1973),

$$P_L(t) \approx P_\infty(t) f(L/\xi_\infty(t); u/L^{\Theta/\nu}), \tag{1}$$

where, for $d > 4$, $\Theta = \frac{1}{2}(d-4)$ is the leading correction-to-scaling exponent at the Gaussian fixed point and $P(t)$ is a general, critical property with subscript L referring to a finite system and ∞ referring to the infinite, Gaussian system. If u could be set equal to zero in the expression (1), then we would recover the usual finite-size scaling form (albeit with the *Gaussian*, and not necessarily mean field, exponents). however, for $d > d_c$, u is normally a 'dangerous' irrelevant variable which may make $f(x; z)$ singular as $z \to 0$. In general, we consider two different limits: (A) $x \to \infty$, $z \to 0$, and (B) $x \to 0$, $z \to 0$.

In the limit (A), the effect of finite-size is not present and $P_L(t) \to P_0(t)$ where $P_0(t)$ denotes the infinite volume limit of P with the *mean field* exponents. In order

for this to hold, $f(x; z) \sim x^a z^b$ where a, b are such that the L-dependence is cancelled and t-dependence reduces to the mean-field case. For example, for the specific heat, $a = 4 - d$, $b = -1$, and for the susceptibility $a = b = 0$. In either case, the deviation of $P_L(t)$ from $P_0(t)$ should be asymptotically small.

In case (B), we again have $f(x; z) \sim x^{a'} z^{b'}$; however, we should now have *no* residual t-dependence. This requirement fixes a', but the value of b' is not obvious and in general has to be calculated explicitly. For the case of the susceptibility (in the totally finite geometry), the calculation of Brézin (1982) indicates $b' = -\frac{1}{2}$. Thus

$$\chi_L(T_c) \sim L^2 \times L^{(d-4)/2}, \tag{2}$$

which is part of his equation (59). The so-called 'failure' of finite-size scaling refers to the presence of the correction factors, such as $L^{(d-4)/2}$ in equation (2) that arise from the singular behaviour of the scaling function†.

Now, going back to the scaling statement (1), we see that the *shift* in the critical point is in general given by

$$t_c(L) \sim L^{-1/\nu} \bar{f}(u/L^{\Theta/\nu}). \tag{3}$$

If one wishes to obtain the behaviour of the shift as a function of L, the question is whether $\bar{f}(z)$ has a singular behaviour as $z \to 0$. For the sake of argument, suppose it did, and let $\bar{f}(z) \sim z^c$. Thus, if $c > 0$, $x_c(L) \equiv L t_c(L)^\nu \to 0$ as $L \to \infty$, while if $c < 0$, $x_c(L) \to \infty$ as $L \to \infty$. Hence, $c > 0$ corresponds to case (B) for t in the neighbourhood of $t_c(L)$, and $c < 0$ corresponds to case (A).

In the first case, I have already argued that there is asymptotically *no* t-dependence for $P_L(t)$. Since the neighbourhood of $t_c(L)$ is the region of sharpest t-dependence, this leads to a contradiction. In the second case ($c < 0$), the argument is a bit more subtle. First, if the shifted critical point is, say, a true divergence (such as for the susceptibility of an Ising system with two or more *infinite* dimensions), then this possibility is also clearly ruled out. This is because $P_L(t_c(L))$ is divergent while $P_0(t_c(L))$ is non-singular and at the same time in case (A) the deviation of $P_L(t)$ from $P_0(t)$ should be asymptotically small. Secondly, if the shifted 'critical' point is, say, a rounded maximum (such as for a totally finite system), similar arguments can be applied to the derivatives of $P_L(t)$, comparing these to the derivatives of $P_0(t)$, at $t = t_c(L)$ as $L \to \infty$. In general, in case (A), $P_L(t)$ must behave asymptotically *like* $P_0(t)$, but this is impossible at $t = t_c(L)$. Thus, the case $c < 0$ also leads to a contradiction. Hence, I argue that $\bar{f}(z)$ does not have a singularity, in contrast to $f(x; z)$ of equation (1), and

$$t_c(L) \sim (\text{constant}) L^{-1/\nu}. \tag{4}$$

Therefore, the finite-size scaling of the relevant, thermal variable is unaltered. Similarly, the fluctuation in measured values of $t_c(L)$ should also scale as (constant)$L^{-1/\nu}$. Thus, finite-size scaling is violated for $d > d_c$ *only* in a very specific sense (cf equation (2)).

A consistency check on equation (4) can be made based on the work of Fisher *et al* (1973). They used finite-size scaling arguments which essentially incorporate‡

† This mechanism can also be invoked to explain why the fractal dimension of the percolating cluster (at p_c) remains 4 for $d > d_c = 6$ (Aharony *et al* 1983; see also Alexander *et al* 1984).

‡ Fisher *et al* (1973) in fact used the relation $\theta = 1/\nu$ where θ is the 'rounding' exponent (Fisher 1972). However, an extension to their arguments by replacing $i (\propto t - t_c(L))$ by t in the spirit of equation (1) also identifies θ with the 'shift' exponent (Fisher 1972).

equation (4) and found a scaling law for the helicity modulus exponent v,

$$v = 2 - \alpha - 2\nu, \tag{5}$$

for all d. Equation (5) is consistent with all known results, and in particular, for the ideal Bose gas and spherical model for $d > 4$, we have the exact values,

$$v = 1, \qquad \alpha = 0, \qquad \nu = \tfrac{1}{2}. \tag{6}$$

Because of the specific sense of the 'failure' of finite-size scaling for $d > d_c$, it also becomes imperative to clarify statement (a) of Sahimi and Jerauld (1983): what is the precise relationship between large-cell renormalisation and finite-size scaling? Unfortunately, the answer to this question is not known, and evidently it must depend on what particular large-cell scheme is used. The only *known* connection is between a hypothetically *exact* renormalisation and some form of finite-size scaling (Brézin 1982, Suzuki 1977). Even this type of connection requires certain further assumptions not originally needed for renormalisation (Privman 1983).

Sahimi and Jerauld (1983) claim that Reynolds *et al* (1980) have shown the 'equivalence' of the large-cell approach to finite-size scaling. Let us consider this point more carefully. In their study of the percolation problem, Reynolds *et al* (1980) made an observation that their eigenvalue λ_b and the fluctuation in the measured value of p_c, $\sigma_b \equiv \langle (p_c(b) - \langle p_c(b) \rangle)^2 \rangle_b^{1/2}$ appear to satisfy asymptotically for large b

$$\lambda_b \sim (\text{constant})/\sigma_b. \tag{7}$$

On the other hand, their large-cell renormalisation can be written as

$$1/\nu = \ln \lambda_b / \ln b + f(b)/\ln b, \tag{8}$$

where the second term must vanish as $b \to \infty$ for their scheme to give correct ν. If we assume $f(b) \sim$ constant, then (7) and (8) would imply finite-size scaling for σ_b:

$$\sigma_b \sim (\text{constant})\, b^{-1/\nu}. \tag{9}$$

Thus, the two assumptions together do imply a *particular* form of finite-size scaling†; however, equation (9) is a type of finite-size scaling for the percolation analogue of the thermal variable, p, which we have already argued does not 'fail' even for $d > d_c$. In addition, even if equation (9) did fail, it could be due to the failure of equation (7) or to the assumption $f(b) \sim$ constant. (In particular, behaviour such as $f(b) \sim \ln \ln b$ would alter (9) without invalidating the renormalisation scheme.)

Lastly, I shall present some new numerical results on the random walk in $d = 3$ and 4 and show the reversal of the initial trend observed by Sahimi and Jerauld (1983) (statement (c)) which evidently helped convince them of their conclusion. Their cell renormalisation consists of counting *all* spanning walks (of lengths less than a certain prescribed limit) starting from a corner site in the hypercubic cell. Thus, they have in general

$$K' = \sum_n c_n K^n, \tag{10}$$

where c_n is the number of such walks of n steps and K is the step fugacity. This

† In a more superficial sense, equation (8) alone with the assumption of $f(b) \sim$ constant gives a relation resembling finite-size scaling: $\lambda_b \sim (\text{constant})\, b^{1/\nu}$. However, to establish a true connection with finite-size scaling, one must relate λ_b to a *physical* quantity as Reynolds *et al* (1980) have done for percolation by equation (7).

method was developed by Family and Gould (1983) and already described by Gould *et al* (1983), and like the renormalisation of some other models discussed there, it becomes computationally unfeasible even for a relatively small cell size b. In this case, however, this problem is trivially avoided by rewriting equation (10) as

$$\bar{K}'/z = \sum_n (c_n/z^n)\bar{K}^n, \tag{11}$$

where z is the lattice coordination number and $\bar{K} \equiv zK$, $\bar{K}' \equiv zK'$ are the *kinetic* step fugacities introduced by Nakanishi and Family (1984b). Once this is done, each coefficient in equation (11) can be estimated by Monte Carlo sampling, and this makes computation for much larger b quite feasible.

My estimates obtained by this method (using the Sahimi–Jerauld prescription for the length cutoff) are shown in table 1 for $d = 3$ and table 2 for $d = 4$. In either case, the reversal of the initial trend is quite plainly observable. Random walks can be considered as a growth process (as well as an 'equilibrium' phenomenon), and indeed this sort of non-monotonic convergence appears to be quite general for many kinetic

Table 1. Cell renormalisation estimates of ν and the connective constant μ for the random walk in $d = 3$. The results for cell sizes $b = 2, \ldots, 5$ were calculated by Sahimi and Jerauld (1983) from their closed-form recursion relations (indicated by †). Error bars indicate one standard deviation.

b	2	3	4	5	10	20	30	50
ν	0.5236† 0.5242 ± 0.0010[a]	0.4818†	0.4664†	0.4423†	0.446 ± 0.008[b]	0.473 ± 0.011[b]	0.474 ± 0.014[c]	0.476 ± 0.012[d]
μ	4.1356† 4.154 ± 0.016[a]	4.7619†	5.0890†	5.1312†	5.737 ± 0.011[b]	5.904 ± 0.010[b]	5.949 ± 0.008[c]	5.979 ± 0.003[d]

[a] Based on three data sets totalling 5×10^5 (spanning and non-spanning) walks.
[b] Based on five data sets totalling 5×10^6 walks.
[c] Based on seven data sets totalling 1.65×10^7 walks.
[d] Based on five data sets totalling 5×10^7 walks.

Table 2. Cell renormalisation estimates of ν and μ for the random walk in $d = 4$. The estimates for ν for $b = 2$, 3 were given by Sahimi and Jerauld (1983) (indiated by †) while the connective constant μ for $b = 2$ is computed from their recursion relation. Error bars indicate one standard deviation.

b	2	3	10	15	20
ν	0.4813† 0.4815 ± 0.0004[a]	0.4438†	0.448 ± 0.012[b]	0.451 ± 0.010[c]	0.465 ± 0.010[c]
μ	5.3883 5.389 ± 0.005[a]	—	7.581 ± 0.034[b]	7.773 ± 0.015[c]	7.845 ± 0.015[c]

[a] Based on three data sets totalling 1.2×10^6 (spanning and non-spanning) walks.
[b] Based on five data sets totalling 5×10^6 walks.
[c] Based on three data sets totalling 5×10^7 walks.

growth processes (Nakanishi and Family 1984b). This may indicate some structural crossover at intermediate length scales, and may possibly hold a key to understanding these growth processes.

In summary, I have presented a critique of the argument that large-cell renormalisation is invalid for $d > d_c$. The conclulsion is that the method is still valid for $d > d_c$ as long as it does not reduce to the use of finite-size scaling to calculate certain properties of the scaling function which are affected by the dangerous singularity found by Brézin (1982). In particular, the calculation of the correlation length exponent ν should be valid. I demonstrate this explicitly by large-cell renormalisation on the random walk for $d = 3, 4$ which show a reversal of the trend cited by Sahimi and Jerauld (1983) as evidence for the claim that the method failed.

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